On future asymptotics of polarized Gowdy $\mathbb{T}^3$-models

Thomas Jurke

Max-Planck-Institut für Gravitationsphysik
Am Mühlenberg 1
14476 Golm[*]

February 7, 2008

Abstract

Gowdy's model of cosmological spacetimes is a much investigated subject in classical and quantum gravity. Depending on spatial topology recollapsing as well as expanding models are known. Several analytic tools were used in order to clarify singular behaviour in this class of spacetimes. Here we investigate the structure of a certain subclass, the polarized Gowdy models with spatial $\mathbb{T}^3$-topology, in the large. The asymptotics for general solutions of the dynamical equation for one of the gravitational degrees of freedom plays a key role while the asymptotic behaviour of the remaining metric function is a result of solving the Hamiltonian constraint equation. Using both we are able to prove (future) geodesic completeness in all spacetimes of this type.

1 Introduction

The cosmological spacetime model by Gowdy (1974) has experienced a great deal of attention over the last years. These models are especially interesting because they allow rigorous analytical investigations in inhomogeneous spacetimes. Various spatial topologies are allowed but – as is mostly done – we will concentrate on vacuum spacetimes with $\mathbb{R}_+ \times \mathbb{T}^3$-topology. A basic analytical tool for this class of spacetimes was provided by Moncrief (1981) who proved existence of global smooth solutions for the corresponding Einstein equations. Investigations concerning the singular asymptotic behaviour were subjects of interest almost exclusively. Now, we have a quite clear picture what happens in the polarized as well as the full Gowdy $\mathbb{T}^3$-model (see Isenberg, Moncrief (1990), Rendall (2000),

---

[*] current address: Weierstraß-Institut für Angewandte Analysis und Stochastik, Mohrenstraße 39, 10117 Berlin

Ringström (2002) and references therein). On the other hand, due to Chruściel, Isenberg and Moncrief (1990) there exist some statements about the asymptotic behaviour for large times in the case of polarized spacetimes. In this paper we prove such theorems, describing the long time asymptotics for general polarized Gowdy models and using this results to prove future completeness of any causal geodesic.

2 The polarized model

By definition a smooth, orientable spacetime is a cosmological model of Gowdy type if it is a maximally extended, globally hyperbolic solution of the vacuum Einstein equations having compact Cauchy surfaces and a pseudo-Riemannian metric, invariant under an effective $U(1) \times U(1)$ group action on these Cauchy surface. This action is generated by two spacelike Killing vector fields which are assumed to have vanishing twist constants. A Gowdy model is called polarized if the defining Killing vector fields are everywhere mutually orthogonal. By virtue of this definition we may describe a polarized Gowdy metric by the line element

$$\mathrm{d}s^2 \;=\; \mathrm{e}^{2a(t,x)}\left(-\mathrm{d}t^2 + \mathrm{d}x^2\right) \;+\; t\left(\mathrm{e}^{W(t,x)}\mathrm{d}y^2 + \mathrm{e}^{-W(t,x)}\mathrm{d}z^2\right). \tag{1}$$

The corresponding vacuum Einstein equations are

$$0 \;=\; \frac{\partial^2}{\partial t^2}W \;+\; \frac{1}{t}\frac{\partial}{\partial t}W \;-\; \frac{\partial^2}{\partial x^2}W \tag{2}$$

$$0 \;=\; \frac{\partial^2}{\partial t^2}a \;-\; \frac{\partial^2}{\partial x^2}a \;-\; \frac{1}{4t^2} \;+\; \frac{1}{4}\left[\left(\frac{\partial}{\partial t}W\right)^2 - \left(\frac{\partial}{\partial x}W\right)^2\right] \tag{3}$$

$$0 \;=\; \frac{2}{t}\frac{\partial}{\partial x}a \;-\; \frac{\partial}{\partial t}W\frac{\partial}{\partial x}W \tag{4}$$

$$0 \;=\; \frac{2}{t}\frac{\partial}{\partial t}a \;+\; \frac{1}{2t^2} \;-\; \frac{1}{2}\left[\left(\frac{\partial}{\partial t}W\right)^2 + \left(\frac{\partial}{\partial x}W\right)^2\right]. \tag{5}$$

As in the general non-polarized case equation (3) is always satisfied as a consequence of the others. Thus, we have essentially to deal with a single linear equation of second order while function a is the integral of the constraint equation (5).

3 On the asymptotics for solutions of the Euler-Poisson-Darboux equation

Now, we start by investigating equation (2). As in literature we will sometimes call this equation "Euler-Poisson-Darboux equation". Following Mon-

crief (1981) it will be sufficient to have only functions $W = W(t, x)$ of a certain regularity in mind which are defined for all $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^1$.

As a first result we get asymptotic homogenity for all solutions of (2). Namely, we will show that for large t the mean value,

$$
\bar{W}(t) \quad \underset{\mathrm{Df}}{=} \quad \frac{1}{2\pi} \int\limits_0^{2\pi} W(t, x) \, \mathrm{d}x, \tag{6}
$$

will dominate while the x-dependent terms show a $t^{-\frac{1}{2}}$ fall-off behaviour.

Theorem 1 *Let $W \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1; \mathbb{R})$ be a solution of Euler-Poisson-Darboux equation (2) and $\bar{W}$ as decleared in (6). Then $\bar{W}$ solves (2), too.*

Proof: For every $t \in \mathbb{R}_+$ function W as well as its first and second derivative are continuous, hence $\mathbb{T}^1$-integrable and $\bar{W}$ is well-defined. Since W is sufficiently smooth $\bar{W}$ depends on its parameter $t \in \mathbb{R}_+$ smoothly, hence $\bar{W} \in \mathcal{C}^2(\mathbb{R}_+; \mathbb{R})$ and

$$
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bar{W}(t) \quad = \quad \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} \frac{\partial}{\partial t}W(t, x) \, \mathrm{d}x \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial^2}{\partial t^2}W(t, x) \, \mathrm{d}x.
$$

Now it is easy to integrate (2)

$$
\frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\partial^2}{\partial t^2}W \, \mathrm{d}x \quad + \quad \frac{1}{2\pi t}\int\limits_0^{2\pi} \frac{\partial}{\partial t}W \, \mathrm{d}x \quad = \quad \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\partial^2}{\partial x^2}W \, \mathrm{d}x
$$

$$
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bar{W} \quad + \quad \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\bar{W} \quad = \quad \frac{1}{2\pi}\frac{\partial W}{\partial x}\bigg|_0^{2\pi}
$$

and since the derivative of smooth periodic functions have same periods we get

$$
\frac{\partial^2}{\partial t^2}\bar{W} \quad + \quad \frac{1}{t}\frac{\partial}{\partial t}\bar{W} \quad = \quad 0 \quad \overset{!}{=} \quad \frac{\partial^2}{\partial x^2}\bar{W}.
$$

$\blacksquare$

The general solution of (2) which does not depend on x is

$$
\bar{W}(t) \quad = \quad \gamma \; + \; \beta \cdot \ln t, \tag{7}
$$

where β and γ may be fixed by some initial values of $\bar{W}$ and $\bar{W}_t$ at $t_0 > 0$. Due to the linearity of (2) ψ in

$$
t^{-\frac{1}{2}}\psi(t, x) \quad \underset{\mathrm{Df}}{=} \quad W(t, x) - \bar{W}(t)
$$

3

is unique and solves (2), too. Obviously this newly defined function ψ, $\psi \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1; \mathbb{R})$, has zero mean value for every $t \in \mathbb{R}_+$

$$\bar{\psi}(t) \underset{\text{Df}}{=} \frac{1}{2\pi} \int_0^{2\pi} \psi(t,x) \, \mathrm{d}x = 0 \tag{8}$$

and satisfies

$$\frac{\partial^2}{\partial t^2}\psi - \frac{\partial^2}{\partial x^2}\psi = -\frac{1}{4t^2}\psi. \tag{9}$$

Now, in order to prove that any solution of this equation has to be bounded we need the following estimate as prerequisite. Here we have formulated the lemma in dependence of a parameters $t \in \mathbb{R}_+$, since this is the form we will need.

Lemma 2 *Let $f: \mathbb{R}_+ \times \mathbb{T}^1 \to \mathbb{R}$ be continuously differentiable and*

$$\frac{1}{2\pi} \int_0^{2\pi} f(t,x) \, \mathrm{d}x = 0$$

for every t. Then we have for each $(t,x) \in \mathbb{R}_+ \times \mathbb{T}^1$:

$$|f(t,x)|^2 \leq 2\pi \int_0^{2\pi} \left| \frac{\partial}{\partial x} f(t,x) \right|^2 \mathrm{d}x \equiv 2\pi \, \|f_x(t,.)\|_{L^2}^2$$

Proof: The mean value theorem of integral calculus for continuous functions gives us to any t some $x_0 \in \mathbb{T}^1$ with $f(t, x_0) = 0$. Furthermore $|x - x_0| \leq 2\pi$ is true for each $x \in \mathbb{T}^1$. Using Schwartz' inequality we get

$$|f(t,x)| = \left| \int_{x_0}^{x} \frac{\partial}{\partial \xi} f(t,\xi) \, \mathrm{d}\xi \right| \leq \int_0^{2\pi} \left| \frac{\partial}{\partial x} f(t,x) \right| \mathrm{d}x$$

$$\leq \left(\int_0^{2\pi} 1^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \cdot \left(\int_0^{2\pi} \left| \frac{\partial}{\partial x} f(t,x) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}}$$

that is

$$|f(t,x)|^2 \leq 2\pi \int_0^{2\pi} \left| \frac{\partial}{\partial x} f(t,x) \right|^2 \mathrm{d}x.$$

∎

Now we are able to show boundedness of ψ, uniformly on every strictly positive intervall. The key element in the proof is the investigation of a certain energy functional corresponding to (9).

4

Theorem 3 *To every function $\psi \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1; \mathbb{R})$, satisfying both (8) and (9), there is a positive constant C_{t_0} depending only on $t_0 > 0$, such that for every $t \geq t_0$ and all $x \in \mathbb{T}^1$ the inequality*

$$|\psi(t, x)| < C_{t_0} \tag{10}$$

holds.

Remark 4 *One verifies that equation (9) has unbounded solutions, too. But none of them satisfies (8) which underlines the importance of this assumption.*

Proof: Let us define the energy

$$\varepsilon(t) \underset{\text{Df}}{=} \frac{1}{2} \int_0^{2\pi} \left({\psi_t}^2 + {\psi_x}^2\right) \mathrm{d}x.$$

Our smoothness assumptions in ψ guarantees differentiability of ε in $\mathbb{R}_+$. Integrating by parts and using equation (9) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon = -\frac{1}{4t^2} \int_0^{2\pi} \psi_t \psi \, \mathrm{d}x.$$

Since $\frac{1}{2}({\psi_t}^2 + \psi^2) + \psi_t \psi \geq 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon \leq \frac{1}{4t^2} \cdot \frac{1}{2} \int_0^{2\pi} \left({\psi_t}^2 + \psi^2\right) \, \mathrm{d}x.$$

Now, lemma 2 permits an estimate of the left hand side of the inequality in terms of ε itself:

$$\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon \leq \frac{\pi^2}{t^2}\varepsilon$$

Since $t \geq t_0 > 0$ we find

$$\varepsilon(t) \leq \varepsilon(t_0) + \pi^2 \int_{t_0}^{t} s^{-2} \varepsilon(s) \, \mathrm{d}s$$

and Gronwall's inequality gives the uniform bound

$$\varepsilon(t) \leq \varepsilon(t_0) + \varepsilon(t_0) \left(e^{\frac{\pi^2}{t_0} - \frac{\pi^2}{t}} - 1\right) < \varepsilon(t_0) \, e^{\frac{\pi^2}{t_0}}.$$

Hence, the energy of an arbitrary solution is for every $t > t_0$ bounded and by definition non-negative. Now, up to a factor we can control the $L^2(\mathbb{T}^1)$-norm

of ψ_x for all $t \geq t_0$ by the square root of the same constant and since ψ satisfies all suppositions of lemma 2 a short calculation yield finally

$$|\psi(t,x)| \;<\; \sqrt{4\pi\,\varepsilon(t_0)}\; \mathrm{e}^{\frac{\pi^2}{2t_0}} \;\equiv\; C_{t_0}$$

for all $t \geq t_0 > 0$ and all $x \in \mathbb{T}^1$, hence C_{t_0} fulfills all needs. $\blacksquare$

We have shown that every spatially periodic solution of the Euler-Poisson-Darboux equation can be cast into the form

$$W(t,x) \;=\; \gamma \,+\, \beta \cdot \ln t \,+\, t^{-\frac{1}{2}}\psi(t,x)$$

where ψ is bounded in $t \in [t_0,\infty)$. On the other hand for applications in the following sections we need deeper insights into the asymptotic behaviour of ψ. Using the boundedness of ψ we can show that solutions to

$$\frac{\partial^2}{\partial t^2}\psi \;-\; \frac{\partial^2}{\partial x^2}\psi \;=\; -\frac{1}{4t^2}\psi$$

behave like solutions of the free wave equation in two dimensions with a remainder falling off in time.

Theorem 5 *Let $\psi \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1; \mathbb{R})$ be a bounded solution of*

$$\frac{\partial^2}{\partial t^2}\psi \;-\; \frac{\partial^2}{\partial x^2}\psi \;=\; -\frac{1}{4t^2}\psi, \tag{11}$$

with the side condition

$$\frac{1}{2\pi}\int_0^{2\pi} \psi(t,x)\,\mathrm{d}x \;=\; 0$$

for every $t \in \mathbb{R}_+$. Then there exist uniquely defined functions $\nu \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1; \mathbb{R})$ and $\omega \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1; \mathbb{R})$ as well as a constant C depending only on the choise of t_0 such that

$$\psi(t,x) \;=\; \nu(t,x) \,+\, \omega(t,x)$$

$$\frac{\partial^2}{\partial t^2}\nu \;=\; \frac{\partial^2}{\partial x^2}\nu$$

$$|\omega(t,x)| \;\leq\; C \cdot t^{-1}$$

$$|\omega_t(t,x)| \;\leq\; C \cdot t^{-1}$$

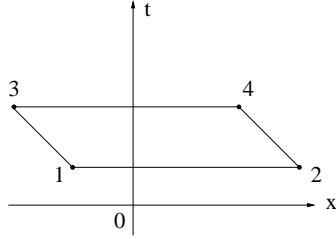*for all $t \in [t_0,\infty)$, $t_0 > 0$ and all $x \in \mathbb{T}^1$.*

Proof: Let

$$u \;\underset{\mathrm{Df}}{=}\; t \,+\, x$$

$$v \;\underset{\mathrm{Df}}{=}\; t \,-\, x.$$
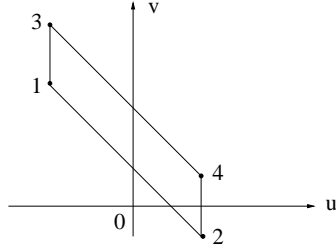
6

In this coordinates (11) reads

$$\frac{\partial^2}{\partial u \, \partial v} \psi \;\; = \;\; -\frac{1}{4}\frac{\psi}{(u+v)^2}. \tag{12}$$

As a consequence of this transformation some parallelogram, say $\overline{1234}$

$$\left(t^{(1)} \equiv t_0, \qquad x^{(1)} \equiv x_0 = u_0 - t_0\right)$$
$$\left(t^{(2)} = t_0, \qquad x^{(2)} = x_0 + 2\pi\right)$$
$$\left(t^{(3)} = t_0 + U, \quad x^{(3)} = x_0 - U\right)$$
$$\left(t^{(4)} = t_0 + U, \quad x^{(4)} = x_0 + 2\pi - U\right)$$

has new vertices as follows:

$$\left(u^{(1)} \equiv u_0, \qquad v^{(1)} \equiv v_0 = 2t_0 - u_0\right)$$
$$\left(u^{(2)} = u_0 + 2\pi, \quad v^{(2)} = v_0 - 2\pi\right)$$
$$\left(u^{(3)} = u_0, \qquad v^{(3)} = v_0 + 2U\right)$$
$$\left(u^{(4)} = u_0 + 2\pi, \quad v^{(4)} = v_0 - 2\pi + 2U\right)$$

The periodicity in x-space is the identity

$$(u, v) \;\; \equiv \;\; (u + 2\pi n, v - 2\pi n) \tag{13}$$

for all $n \in \mathbb{Z}$ and the over-all condition $t > 0$ converts to $u + v > 0$. Both are generally assumed in what follows. – The estimate

$$|\psi_u(u, v_2) \;-\; \psi_u(u, v_1)| \;\; \le \;\; \int_{v_1}^{v_2} \frac{|\psi| \, \mathrm{d}v}{(u+v)^2} \;\;\overset{(10)}{\le}\;\; C_{t_0}\left(\frac{1}{u+v_1} - \frac{1}{u+v_2}\right)$$

yields boundedness of $|\psi_u(u, v)|$ for any u while $v \to \infty$. But the same inequality also proves with $v_2 = v_1 + \Delta$, $\Delta > 0$, the existence of a finite limit $F'(u)$ for any u, hence

$$F'(u) \;\; \underset{\mathrm{Df}}{=} \;\; \psi_u(u, 2t_0 - u) + \lim_{U \to \infty} \int_{2t_0 - u}^{2t_0 - u + 2U} \psi_{uv}(u, v) \, \mathrm{d}v \tag{14}$$

7

is well-defined. Periodicity of $F'(u)$ is due to (13) along with

$$
\begin{aligned}
F'(u + 2\pi) &= \psi_u(u + 2\pi, 2t_0 - u - 2\pi) - \frac{1}{4} \int\limits_{2t_0-u-2\pi}^{\infty} \frac{\psi(u + 2\pi, v)}{(u + 2\pi + v)^2} \, dv \\
&= \psi_u(u, 2t_0 - u) - \frac{1}{4} \int\limits_{2t_0-u}^{\infty} \frac{\psi(u + 2\pi, \tilde{v} - 2\pi)}{(u + \tilde{v})^2} \, d\tilde{v} \\
&= F'(u).
\end{aligned}
$$

Another property of $F'(u)$ is its zero mean value. To show this we transform the defining integral back to t-x-coordinates according to the scetches from the beginning of this proof. Let (u_0, v_0), that is (t_0, x_0), $t_0 > 0$, an arbitrary point in characteristic or space-time coordinates, respectively. Then

$$
\int\limits_{u_0}^{u_0+2\pi} F'(u) \, du = - \int\limits_{u_0}^{u_0+2\pi} \lim_{U \to \infty} \int\limits_{2t_0-u}^{2t_0-u+2U} \frac{\psi(u, v)}{4(u + v)^2} \, dv \, du \tag{15}
$$

$$
= \lim_{U \to \infty} \int\limits_{t_0}^{t_0+U} \frac{1}{2t^2} \int\limits_{x_0+t_0-t}^{x_0+t_0-t+2\pi} \psi(t, x) \, dx \, dt = 0, \tag{16}
$$

since the term under the integral is zero for every t. A direct consequence is the periodicity of primitives. Let

$$
\tilde{F}(u) \underset{\text{Df}}{=} \int F'(u) \, du
$$

such an (arbitrary but fixed) primitive. Then the function

$$
F(u) \underset{\text{Df}}{=} \tilde{F}(u) - \frac{1}{2\pi} \int\limits_{u_0}^{u_0+2\pi} \tilde{F}(u) \, du
$$

is defined uniquely and especially

$$
\int\limits_{u_0}^{u_0+2\pi} F(u) \, du = 0 \tag{17}
$$

is always satisfied. Now, starting with the inequality

$$
|\psi_v(u_2, v) - \psi_v(u_1, v)| \leq \int\limits_{u_1}^{u_2} \frac{|\psi| \, du}{(u + v)^2} \leq C_{t_0} \left(\frac{1}{u_1 + v} - \frac{1}{u_2 + v} \right)
$$

we construct a likewise uniquely defined function $G(v)$, where all steps follow as above endowing G with corresponding properties. Let us finally introduce the functions:

$$\nu(u,v) \underset{\text{Df}}{=} F(u) + G(v) \tag{18}$$

$$\omega(u,v) \underset{\text{Df}}{=} \psi(u,v) - \nu(u,v) \tag{19}$$

Obviously ν and ω are uniquely defined for any fixed ψ. ν solves the free wave equation $\nu_{uv} = 0$ and regarding the remainder ω we estimate for every u, v subject to $u + v > 0$

$$|\omega_u(u,v)| = |\psi_u(u,v) - \nu_u(u,v)|$$

$$= \left| \psi_u(u,v) - \psi_u(u, 2t_0 - u) - \int_{2t_0 - u}^{\infty} \psi_{uv}(u,v)\, \mathrm{d}v \right|$$

$$= \left| -\int_{v}^{2t_0 - u} \psi_{u\tilde{v}}(u,\tilde{v})\, \mathrm{d}\tilde{v} - \int_{2t_0 - u}^{\infty} \psi_{uv}(u,v)\, \mathrm{d}v \right|$$

$$= \left| \frac{1}{4} \int_{v}^{\infty} \frac{\psi(u,\tilde{v})}{(u+\tilde{v})^2}\, \mathrm{d}\tilde{v} \right| \leq \frac{C_{t_0}}{4(u+v)}, \tag{20}$$

where the estimate follows from (10). In exactly the same way one proves

$$|\omega_v(u,v)| \leq \frac{C_{t_0}}{4(u+v)}.$$

Now, since

$$|\omega_t(t,x)| = |\omega_u(u,v) + \omega_v(u,v)| \leq \frac{C_{t_0}}{2(u+v)} = \frac{C_{t_0}}{4t} \tag{21}$$

$$|\omega_x(t,x)| = |\omega_u(u,v) - \omega_v(u,v)| \leq \frac{C_{t_0}}{2(u+v)} = \frac{C_{t_0}}{4t} \tag{22}$$

for all $t \in [t_0, \infty)$, $x \in \mathbb{T}^1$ we have

$$\|\omega_x(t,.)\|_{L^2}^2 = \int_{0}^{2\pi} \omega_x{}^2\, \mathrm{d}x \leq \frac{\pi C_{t_0}{}^2}{8t^2}.$$

Corollary 9 below proves the properties of ω

$$\omega(t, x + 2\pi) = \omega(t,x)$$

$$\int_{0}^{2\pi} \omega(t,x)\, \mathrm{d}x = 0$$

9

for every $(t,x) \in \mathbb{R}_+ \times \mathbb{T}^1$. Hence all assumptions of lemma 2 are satisfied which ensures the existence of some constant $C \underset{\text{Df}}{=} \frac{\pi}{2} C_{t_0} > 0$ with

$$|\omega(t,x)| \quad \leq \quad \frac{C}{t} \tag{23}$$

for all $t \in [t_0, \infty)$, $t_0 > 0$ and all $x \in \mathbb{T}^1$. ∎

The existence and uniqueness in the definition of function ν allow us to introduce the following term.

Definition 6 *The function ν is called the ψ-associated solution of the free wave equation, if $\psi \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1; \mathbb{R})$ solves*

$$\frac{\partial^2}{\partial t^2}\psi - \frac{\partial^2}{\partial x^2}\psi \quad = \quad -\frac{1}{4t^2}\psi$$

with the side condition

$$\int_0^{2\pi} \psi(t,x) \, \mathrm{d}x \quad = \quad 0$$

for every $t \in [t_0, \infty)$ and $\nu \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1; \mathbb{R})$ is defined as in equation (18).

Here follows a series of corollaries establishing properties of ν as well as ω which were used in the previous proof or are of frequent use in the sections below.

Corollary 7 *Let ν be the ψ-associated solution of the free wave equation and $t_0 > 0$. Then there exists a positive constant C, such that for all $t \in [t_0, \infty)$ and $x \in \mathbb{T}^1$ the functions ν, ν_t and ν_x are bounded:*

$$\max_{\substack{t \geq t_0 > 0 \\ x \in \mathbb{T}^1}} \{|\nu(t,x)|, |\nu_t(t,x)|, |\nu_x(t,x)|\} \quad < \quad C \tag{24}$$

Further, for all $n \in \mathbb{Z}$ are:

$$\nu(t, x + 2\pi n) \quad = \quad \nu(t,x) \quad = \quad \nu(t + 2\pi n, x) \tag{25}$$

$$\nu_t(t + 2\pi n, x) \quad = \quad \nu_t(t,x) \tag{26}$$

$$\nu_x(t + 2\pi n, x) \quad = \quad \nu_x(t,x) \tag{27}$$

$$\int_0^{2\pi} \nu(t,x) \, \mathrm{d}x \quad = \quad 0 \tag{28}$$

Proof:

10

(24) By definition it is $\nu(u,v) = F(u) + G(v)$. F is periodic with vanishing mean value. W.l.o.g. is $F(u_0) = 0$, $\psi_u(u, v_\infty) \underset{\mathrm{Df}}{=} \lim\limits_{v \to \infty} \psi_u(u, v)$. Then

$$F(u) \;=\; \int\limits_{u_0}^{u} F'(\tilde{u})\, \mathrm{d}\tilde{u} \;=\; \int\limits_{u_0}^{u} \psi_{\tilde{u}}(\tilde{u}, v_\infty)\, \mathrm{d}\tilde{u} \;=\; \psi(u, v_\infty) \;-\; \psi(u_0, v_\infty).$$

Theorem 3 shows boundedness of ψ (uniformly in the considered domain). The same is true for G. Finally, boundedness of $|\nu_t(t,x)|$ and $|\nu_x(t,x)|$ is a consequence of $|F'(u)|$ and $|G'(v)|$ boundedness established during the proof of theorem 5.

(25) With (17) we have

$$\begin{aligned}
\nu(t + 2\pi n, x) &= F(t + 2\pi n + x) + G(t + 2\pi n - x) \\
&= F(t + x) + G(t - x) \;=\; \nu(t, x)
\end{aligned}$$

and analogously

$$\begin{aligned}
\nu(t, x + 2\pi n) &= F(t + x + 2\pi n) + G(t - x - 2\pi n) \\
&= \nu(t, x).
\end{aligned}$$

(26) Is a direct consequence of (25).

(27) Follows directly from (25).

(28) Property (17) and correspondingly for G yield:

$$\begin{aligned}
\int\limits_0^{2\pi} \nu(t, x)\, \mathrm{d}x \;&=\; \int\limits_t^{t+2\pi} \nu(u, 2t - u)\, \mathrm{d}u \\
&= \int\limits_t^{t+2\pi} (F(u) + G(2t - u))\, \mathrm{d}u \\
&= \int\limits_t^{t+2\pi} F(u)\, \mathrm{d}u + \int\limits_{t-2\pi}^{t} G(v)\, \mathrm{d}v \;=\; 0
\end{aligned}$$

∎

Corollary 8 *Let ν be the ψ-associated solution of the free wave equation. Then the following are equivalent:*

$$\psi(t, x) \;\equiv\; 0 \tag{29}$$

$$F(t + x) \;\equiv\; 0 \quad and \quad G(t - x) \;\equiv\; 0 \tag{30}$$

$$\nu(t, x) \;\equiv\; 0 \tag{31}$$

11

Proof:

$(29) \Rightarrow (30)$: F is constant due to (14), from (17) follows the statement with respect to F; analogously for G.

$(30) \Leftrightarrow (31)$: ($\Rightarrow$) is due to (18) and ($\Leftarrow$) due to (18) and (17).

$(31) \Rightarrow (29)$: It is $\psi(t,x) = \omega(t,x)$. The estimate (22) reads here in a first step

$$|\psi_x(t,x)|_1 \quad \leq \quad \frac{C_{t_0}}{4t}$$

and according to lemma 2

$$|\psi(t,x)|_1 \quad \leq \quad \frac{\pi C_{t_0}}{2t}.$$

Now, this improved estimate can be used iteratively in (20), leading to an n-th order estimate

$$|\psi(t,x)|_n \quad \leq \quad \left(\frac{\pi}{2}\right)^n \frac{C_{t_0}}{n!} t^{-n}$$

for every $x \in \mathbb{T}^1$ and every $t \in [t_0, \infty)$. Due to (21) the same estimate is valid for ψ_t. Consequently exists to any $\varepsilon > 0$ an n_0, such that

$$|\psi(t_0, x)|_n \quad < \quad \varepsilon$$
$$|\psi_t(t_0, x)|_n \quad < \quad \varepsilon$$

for all $x \in \mathbb{T}^1$ and all $n \geq n_0$, that is $\psi(t_0, x) = \psi_t(t_0, x) = 0$. Due to the uniqueness of its solutions ψ is the trivial solution of (11). ∎

Corollary 9 *Let ν be the ψ-associated solution of the free wave equation, $\omega \underset{\text{Df}}{=} \psi - \nu$ and $t_0 > 0$. Then there exists a positive constant C, such that for all $t \in [t_0, \infty)$, $x \in \mathbb{T}^1$ and $n \in \mathbb{Z}$:*

$$|\omega(t,x)| \quad < \quad C \tag{32}$$

$$\omega(t, x + 2\pi n) \quad = \quad \omega(t,x) \tag{33}$$

$$\int_0^{2\pi} \omega(t,x) \, \mathrm{d}x \quad = \quad 0 \tag{34}$$

Proof: All properties are direct consequences from the definition of ω, the corresponding properties of ψ along with corollary 7 above. ∎

Corollary 10 *Let ν be the ψ-associated solution of the free wave equation and $\omega \underset{\text{Df}}{=} \psi - \nu$. Then the following is equivalent:*

$$\psi(t,x) \quad \equiv \quad 0 \tag{35}$$

$$\omega(t,x) \quad \equiv \quad 0 \tag{36}$$

12

Proof:

$(35) \Rightarrow (36)$: With corollary 8 is $\nu \equiv 0$.

$(36) \Rightarrow (35)$: It is $\psi \equiv \nu$. Since

$$\frac{\partial^2}{\partial t^2}\psi \; - \; \frac{\partial^2}{\partial x^2}\psi \;\; = \;\; -\frac{1}{4t^2}\psi \;\; \stackrel{!}{\equiv} \;\; 0$$

the statement follows. ∎

Now we summarize the most important features in the asymptotic behaviour of the metric function W. Here we use as remainder $\varkappa \underset{\mathrm{Df}}{=} t^{-\frac{1}{2}}\omega$.

Corollary 11 *Let $W \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1)$ be a real-valued solution of the Euler-Poisson-Darboux equation*

$$\frac{\partial^2}{\partial t^2}W(t,x) \; + \; \frac{1}{t}\frac{\partial}{\partial t}W(t,x) \; - \; \frac{\partial^2}{\partial x^2}W(t,x) \;\; = \;\; 0.$$

Then there exists uniquely defined, bounded, real-valued functions $\nu \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1)$ and $\varkappa \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1)$, constants β, γ, and a positive constant C_{t_0} such that

$$W(t,x) \;\; = \;\; \gamma \; + \; \beta \cdot \ln t \; + \; t^{-\frac{1}{2}}\nu(t,x) \; + \; \varkappa(t,x) \tag{37}$$

$$\frac{\partial^2}{\partial t^2}\nu(t,x) \;\; = \;\; \frac{\partial^2}{\partial x^2}\nu(t,x) \tag{38}$$

and

$$|\varkappa(t,x)| \;\; \leq \;\; C_{t_0} \cdot t^{-\frac{3}{2}} \tag{39}$$

$$|\varkappa_t(t,x)| \;\; \leq \;\; C_{t_0} \cdot t^{-\frac{3}{2}} \tag{40}$$

$$|\varkappa_x(t,x)| \;\; \leq \;\; C_{t_0} \cdot t^{-\frac{3}{2}} \tag{41}$$

for all $t \geq t_0 > 0$ and all $x \in \mathbb{T}^1$.

Proof.   We have already shown (37) and (38), explicitly. But on the other hand estimates (39), (40) and (41) follow directly from the above definition of $\varkappa$ together with (23), (21) and (22), respectively. ∎

4   On the integration of constraints

As mentioned in the beginning part of this paper a special feature of Gowdy's model is the decoupling of the dynamical quantity a. Furthermore, it is well-known that the momentum constraint

$$\frac{\partial}{\partial x}a \;\; = \;\; \frac{1}{2}\,t\,\frac{\partial}{\partial t}W\,\frac{\partial}{\partial x}W \tag{42}$$

is conserved along t developments; that means it is satisfied always if it is satisfied initially.

With this remark in mind we will investigate now the Hamiltonian constraint

$$\frac{\partial}{\partial t}a \;=\; -\frac{1}{4t} \;+\; \frac{1}{4}t\left[\left(\frac{\partial}{\partial t}W\right)^2 + \left(\frac{\partial}{\partial x}W\right)^2\right] \tag{43}$$

exclusively.

Spatially homogeneous spacetimes

According to (7) we find as the general solution of the Euler-Poisson-Darboux equation in spatially homogeneous spacetimes

$$W(t) \;=\; \beta \cdot \ln t \;+\; \gamma.$$

The constraint (43) reads

$$a_t(t) \;=\; \frac{1}{4}(\beta^2 - 1) \cdot t^{-1},$$

leading to

$$a(t) \;=\; \frac{1}{4}(\beta^2 - 1) \cdot \ln t \;+\; \zeta \tag{44}$$

as solution where the arbitrary constant ζ has to be specialized by an initial condition for a.

Not spatially homogeneous spacetimes

The exceptional position of spatially homogeneous among the polarized Gowdy spacetimes is a consequence of the following consideration.

Theorem 12 *Let $W \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1; \mathbb{R})$ be a solution of the Euler-Poisson-Darboux equation. Then there is a positive constant C, such that*

$$|a_t(t,x)| \;\leq\; C$$

for all $t \in [t_0, \infty)$ and all $x \in \mathbb{T}^1$.

Proof: With corollary 11 we have

$$W(t,x) \;=\; \gamma \;+\; \beta \cdot \ln t \;+\; t^{-\frac{1}{2}}\nu(t,x) \;+\; \varkappa(t,x)$$

14

with corresponding properties of ν and estimates for the remainder $\varkappa$; therefore

$$
\begin{aligned}
a_t \;=\;& \frac{1}{4}(\nu_t{}^2 + \nu_x{}^2) \;+\; \frac{1}{2}t^{-\frac{1}{2}}\,\beta\,\nu_t \\
& + \frac{1}{4}t^{-1}\left(\beta^2 - 1 + 2\nu_t\left(t^{\frac{3}{2}}\varkappa_t - \frac{1}{2}\nu\right) + 2\nu_x t^{\frac{3}{2}}\varkappa_x\right) \\
& + \mathcal{O}(t^{-\frac{3}{2}}).
\end{aligned} \tag{45}
$$

The statement follows from corollaries 7 and 11. ∎

Remark 13 *The above equation (45) already reflects the qualitatively differ-
ent asymptotic behaviour in the metric function a depending on the spatial ho-
mogenity or not spatial homogenity of the underlying model. This is so because
corollary 8 along with corollary 10 states that a polarized Gowdy spacetime is
spatially homogeneous if and only if*

$$
\psi \;\equiv\; 0 \;\;\leftrightarrow\;\; \nu \;\equiv\; 0 \;\;\leftrightarrow\;\; \varkappa \;\equiv\; 0.
$$

Since the asymptotics in spatially homogeneous models has been completely
investigated we will in what follows w.l.o.g. assume that $\nu_t = \nu_x = 0$ is valid
not everywhere.

Theorem 14 *Let $a \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1; \mathbb{R})$ be a solution of equation (43) and $\partial_x a = 0$ not everywhere. Then there exist positive constants $\bar{\nu}$ and C along with a
uniquely determined function $\delta \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1; \mathbb{R})$ such that*

$$
a(t,x) \;=\; \bar{\nu}\cdot t \;+\; \delta(t,x)
$$

with

$$
|\delta(t,x)| \;\leq\; C\cdot(1 + t^{\frac{1}{2}})
$$

for every $t \in [t_0, \infty)$, $t_0 > 0$ and every $x \in \mathbb{T}^1$.

Proof: Since a is continously differentiable, there is in $\mathbb{R}_+ \times \mathbb{T}^1$ an open domain
with $a_x \neq 0$ hence $\nu_x \neq 0$. Consequently

$$
\bar{\nu} \;\underset{\mathrm{Df}}{=}\; \frac{1}{8\pi}\int\limits_{t_0}^{t_0+2\pi}(\nu_t{}^2 + \nu_x{}^2)\,\mathrm{d}t \tag{46}
$$

is a positive constant since it is also x-independent due to $\nu_{tt} = \nu_{xx}$ and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\bar{\nu} \;=\;& \frac{1}{4\pi}\int_{t_0}^{t_0+2\pi}(\nu_t\nu_{tx} + \nu_x\nu_{xx})\,\mathrm{d}t \\
=\;& \frac{1}{4\pi}\int_{t_0}^{t_0+2\pi}\nu_x(-\nu_{tt} + \nu_{xx})\,\mathrm{d}t \;+\; \frac{1}{4\pi}\nu_t\nu_x\Big|_{t_0}^{t_0+2\pi} \;=\; 0.
\end{aligned}
$$

15

Let be further

$$\delta_t(t,x) \underset{\text{Df}}{=} -\bar{\nu} + a_t(t,x)$$

$$= -\bar{\nu} + \frac{1}{4}(\nu_t{}^2 + \nu_x{}^2)$$

$$+ \frac{1}{2}t^{-\frac{1}{2}}\beta\,\nu_t$$

$$+ \frac{1}{4}t^{-1}\left(\beta^2 - 1 + 2\nu_t\left(t^{\frac{3}{2}}\varkappa_t - \frac{1}{2}\nu\right) + 2\nu_x t^{\frac{3}{2}}\varkappa_x\right)$$

$$+ \frac{1}{2}t^{-\frac{3}{2}}\beta\left(t^{\frac{3}{2}}\varkappa_t - \frac{1}{2}\nu\right)$$

$$+ \frac{1}{4}t^{-2}\left[\left(t^{\frac{3}{2}}\varkappa_t - \frac{1}{2}\nu\right)^2 + \left(t^{\frac{3}{2}}\varkappa_x\right)^2\right]$$

which is uniquely determined for every fixed solution W of the Euler-Poisson-Darboux equation. So we have

$$\delta(t,x) = \delta(t_0,x) + \frac{1}{4}\int_{t_0}^{t}\left(\nu_\tau{}^2 + \nu_x{}^2\right)\,\mathrm{d}\tau - \bar{\nu}\cdot(t - t_0) + \mathcal{O}(t^{\frac{1}{2}}).$$

Now we consider the second and third term of the right hand side using (46) in detail:

$$\frac{1}{4}\int_{t_0}^{t}\left(\nu_\tau{}^2 + \nu_x{}^2\right)\,\mathrm{d}\tau - \bar{\nu}\cdot(t - t_0)$$

$$= \int_{t_0}^{t}\left[\frac{1}{4}(\nu_\tau{}^2 + \nu_x{}^2) - \frac{1}{8\pi}\int_{t_0}^{t_0+2\pi}(\nu_t{}^2 + \nu_x{}^2)\,\mathrm{d}t\right]\,\mathrm{d}\tau$$

$$= \frac{1}{4}\int_{t_0}^{t_0+2\pi\left[\frac{t-t_0}{2\pi}\right]}(\nu_t{}^2 + \nu_x{}^2)\,\mathrm{d}t - \frac{\left[\frac{t-t_0}{2\pi}\right]}{4}\int_{t_0}^{t_0+2\pi}(\nu_t{}^2 + \nu_x{}^2)\,\mathrm{d}t$$

$$+ \int_{t_0+2\pi\left[\frac{t-t_0}{2\pi}\right]}^{t}\left[\frac{1}{4}(\nu_\tau{}^2 + \nu_x{}^2) - \frac{1}{8\pi}\int_{t_0}^{t_0+2\pi}(\nu_t{}^2 + \nu_x{}^2)\,\mathrm{d}t\right]\,\mathrm{d}\tau$$

Here as usual $[n]$ is the largest integer not greater then n. Now, due to periodicity the first two terms cancel each other. The integrand of the third is bounded, the domain of integration less then 2π so it is

$$\left|\int_{t_0}^{t}\left[\frac{1}{4}(\nu_\tau{}^2 + \nu_x{}^2) - \bar{\nu}\right]\,\mathrm{d}\tau\right| \leq 2\pi C$$

16

and finally

$$|\delta(t,x)| \quad \leq \quad C \cdot (1 + t^{\frac{1}{2}})$$

for a positive constant C, all $x \in \mathbb{T}^1$ and every $t \in [t_0, \infty)$, $t_0 > 0$. ∎

All results concerning the integration of constrains are summarized by the following theorem.

Theorem 15 *Let $W \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1; \mathbb{R})$ be a solution of $W_{tt} + t^{-1}W_t - W_{xx} = 0$. Then any solution $a \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{T}^1; \mathbb{R})$ of*

$$a_t \quad = \quad -\frac{1}{4}t^{-1} \ + \ \frac{1}{4}t\,({W_t}^2 + {W_x}^2)$$

may be cast for all $t \geq t_0 > 0$ and all $x \in \mathbb{T}^1$ into the form

$$a(t,x) \quad = \quad \begin{cases} \frac{1}{4}(\beta^2 - 1)\cdot\ln t \ + \ \zeta, & \text{if } \partial_x a(t,x) \equiv 0 \\ \bar{\nu}\cdot t \ + \ \delta(t,x), & \text{otherwise} \end{cases} \tag{47}$$

where $\bar{\nu}$ is a positive, ζ an arbitrary constant and δ satisfies the inequalities

$$|\delta(t,x)| \quad \leq \quad C \cdot (1 + t^{\frac{1}{2}})$$
$$|\delta_t(t,x)| \quad \leq \quad C$$

with another positive constante C.

5 On geodesic completeness in polarized Gowdy spacetimes

In contrast to the analytical treatments in the first part of this paper we will now investigate some open geometrical questions in polarized Gowdy spacetimes.

Isenberg and Moncrief (1990) established important results with respect to the singular behaviour in this class of spacetimes. However, the question concerning possible singularities in its future development remained open (at least for the case with spatial $\mathbb{T}^3$-topology) and shall be the subject of this section. As usual we will identify the term "singularity" with the term "existence of an uncomplete non-spacelike geodesic" and use the basic concepts and relations of the theory of Lorentzian manifolds and causality without repeating all its definitions. However, for convenience we will summarize some facts for short.

Future will always mean the expanding direction of the universe.

In local coordinates (x^0, x^1, x^2, x^3) the geodesic equation reads:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} \ + \ \sum_{j,k} \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} \quad = \quad 0, \qquad i,j,k = 0,\ldots,3 \tag{48}$$

Among the properties of affinly parametrized geodesics $\mathfrak{x}$ the following are of special importance:

1.) The tangent vector is normalized

$$\sum_{i,j} g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}\tau} \frac{\mathrm{d}x^j}{\mathrm{d}\tau} \;=\; -\varepsilon, \tag{49}$$

where $\varepsilon = 0$, if the geodesic is null and $\varepsilon = 1$, if it is timelike.

2.) The scalar product of any Killing vectorfield X and the geodesic tangent is constant along the curve; in coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \sum_{i,j} g_{ij} X_{\mathfrak{x}}{}^i \frac{\mathrm{d}x^j}{\mathrm{d}\tau} \;=\; 0 \tag{50}$$

Gowdy spacetimes are known to be maximally extended, globally hyperbolic regions which permit foliations by $(t = \text{const})$-hypersurfaces. Consequently, any nonspacelike curve intersects every such hypersurface exactly once. This is important in view of the possibility to take $t \to \infty$ limits along an arbitrary nonspacelike curve. Furthermore, according to our choise of orientation it is

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} \;>\; 0 \tag{51}$$

if τ is as before the parameter of the future directed curve.

Now we can start with the investigations. Since the asymptotic expansion of polarized Gowdy spacetimes is quite different depending on the spatial homogenity/inhomogenity of the model it is necessary to split the proofs accordingly.

Geodesics in spatially homogeneous, polarized spacetimes

Spatially homogeneous, polarized spacetimes correspond to exactly solvable Einstein equations. We had found (7) and (44),

$$W(t) \;=\; \gamma + \beta \cdot \ln t \tag{52}$$

$$a(t) \;=\; \zeta + \frac{1}{4}(\beta^2 - 1) \cdot \ln t, \tag{53}$$

with constants β, γ, ζ. We need the following estimate to prove geodesic completeness in such models.

Lemma 16 *Let* $\left(\mathbb{R}_+ \times \mathbb{T}^3, g = \mathrm{diag}(g_{00}, g_{11}, g_{22}, g_{33})\right)$ *be a spatially homogeneous solution of Einstein's field equation for the polarized Gowdy model. Then there exist a constant* C_β, *such that the Christoffel symbols satisfy for every* $t \in [t_0, \infty)$:

$$\Gamma^0_{11}(t) \;>\; -\frac{1}{2} \, g^{00} \, C_\beta \, t^{-1} \cdot g_{11}$$

18

$$\Gamma_{22}^{0}(t) \quad > \quad -\frac{1}{2} g^{00} C_{\beta} t^{-1} \cdot g_{22}$$

$$\Gamma_{33}^{0}(t) \quad > \quad -\frac{1}{2} g^{00} C_{\beta} t^{-1} \cdot g_{33}$$

Proof: The following section 6 contains the relevant Christoffel symbols. Using (52) and (53) we find:

$$\Gamma_{11}^{0}(t) \quad = \quad -\frac{1}{2} g^{00} \partial_t g_{11} \quad = \quad -\frac{1}{2} g^{00} \cdot \frac{1}{2} (\beta^2 - 1) \, t^{-1} \cdot g_{11}$$

$$\Gamma_{22}^{0}(t) \quad = \quad -\frac{1}{2} g^{00} \partial_t g_{22} \quad = \quad -\frac{1}{2} g^{00} \cdot \quad (\beta + 1) \, t^{-1} \cdot g_{22}$$

$$\Gamma_{33}^{0}(t) \quad = \quad -\frac{1}{2} g^{00} \partial_t g_{33} \quad = \quad -\frac{1}{2} g^{00} \cdot \quad (-\beta + 1) \, t^{-1} \cdot g_{33}$$

The constant C_{β},

$$C_{\beta} \underset{\text{Df}}{=} -\frac{1}{2}(\beta^2 + 3), \tag{54}$$

proves the statement. Observe that $C_{\beta} < 0$. ∎

Theorem 17 *Every inextendable, future directed, nonspacelike geodesic in an arbitrary but spatially homogeneous, polarized Gowdy manifold is in its future direction complete.*

Proof: It is sufficient to regard the t-component of an arbitrary nonspacelike geodesic with affin parameter τ. Due to the assumed homogenity not all of the symbols in section 6 are different from zero. More precisely, the 0-component of (48) reads

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \Gamma_{00}^{0} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \Gamma_{11}^{0} \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + \Gamma_{22}^{0} \left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 + \Gamma_{33}^{0} \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \quad = \quad 0.$$

The causal character of the curve and the choise of time orientation imply (51) and further

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^{-1} \quad =$$

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^{-2} \left[\Gamma_{00}^{0} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \Gamma_{11}^{0} \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + \Gamma_{22}^{0} \left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 + \Gamma_{33}^{0} \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2\right]. \tag{55}$$

Now, using lemma 16 we can estimate the symbols and get

$$\Gamma_{11}^{0} \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + \Gamma_{22}^{0} \left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 + \Gamma_{33}^{0} \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \quad >$$

$$-\frac{1}{2} g^{00} C_{\beta} t^{-1} \left[g_{11} \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + g_{22} \left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 + g_{33} \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2\right].$$

19

On the other hand we have (49)

$$g_{11}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 \; + \; g_{22}\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 \; + \; g_{33}\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \;\; = \;\; -\varepsilon - g_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2$$
$$< \;\; -g_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2$$

where $\varepsilon \in \{0,1\}$ so we get finally $(C_\beta < 0)$

$$\Gamma^0_{11}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 \; + \; \Gamma^0_{22}\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 \; + \; \Gamma^0_{33}\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \;\; > \;\; \frac{1}{2}C_\beta\, t^{-1}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 .$$

Using (55) as well as $\Gamma^0_{00} = \frac{1}{4}(\beta - 1)\cdot t^{-1}$ and (54)

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^{-1} \;\; > \;\; \frac{1}{2}\left[\frac{1}{2}(\beta^2 - 1) + C_\beta\right]t^{-1} \;\; = \;\; -t^{-1}$$

we get an suitable estimate. Integrating twice the result is

$$\tau(t) \;\; > \;\; \tau(t_0) \; + \; C\cdot\ln\frac{t}{t_0}$$

for some positive C. Obviously, by taking $t \to \infty$ we have proved that $\tau \to \infty$ holds along any nonspacelike geodesic. ∎

Geodesics in not spatially homogeneous, polarized space-times

In the case under consideration the components of Gowdy metric are not everywhere x-independent and consequently, the solutions of Einstein equations are not known explicitly. Due to limitation on asymptotic behaviour as well as the more complex form of geodesic equation proofs are more technical here.

We begin by poving some estimates:

Lemma 18 *Let $t_0 > 0$ and $W\colon \mathbb{R}_+ \times \mathbb{T}^1 \to \mathbb{R}$ some smooth solution of $W_{tt} + t^{-1}W_t - W_{xx} = 0$, $W_x \not\equiv 0$. Then there is a positive constant C, such that for all $t > t_0$*

$$\left| -\frac{1}{2t^2} + \frac{1}{2}(W_t{}^2 - W_x{}^2) \right| \;\; \leq \;\; C\cdot t^{-1}$$

holds.

Proof: For t sufficiently large we know according to corollary 11 W is of the form

$$W(t,x) \;\; = \;\; \gamma \; + \; \beta\cdot\ln t \; + \; t^{-\frac{1}{2}}\left(\nu(t,x) \; + \; \omega(t,x)\right). \tag{56}$$

Theorem 3 proved boundedness of $\nu + \omega$ while its partial derivatives are bounded by theorem 5, hence the lemma. ∎

Lemma 19 *For any polarized Gowdy spacetime model there is a constant C_1 and a positive constant C, such that*

$$|g_{ab}| \ + \ |\partial_t g_{ab}| \ + \ |g^{ab}| \ + \ |\partial_t g^{ab}| \quad \leq \quad C \cdot t^{C_1}$$

where $a, b \in \{2, 3\}$ and $t_0 < t < \infty$.

Proof: (56) shows logarithmic growth of W if $\beta \neq 0$, consequently the metric coefficients g_{ab}, its inverses and derivatives can increase at most with power $\beta + 1$. So a constant $C_1 \underset{\mathrm{Df}}{=} |\beta| + 1$ will be sufficient for our needs. ∎

Theorem 20 *Every inextendable, future directed, nonspacelike geodesic in an arbitrary but not spatially homogeneous, polarized Gowdy manifold is in its future direction complete.*

Proof: Let $\mathfrak{x} = (t, x, y, z)$ be the components of some arbitrary nonspacelike geodesic in local coordinates, having affin parameter τ and taking values in an likewise arbitrary, polarized (not spatially homogeneous) Gowdy manifold. As before it will sufficient to restrict considerations on the t-component. Here it holds:

$$
\begin{aligned}
0 \ = \ & \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} \ + \ \Gamma^0_{00} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 \ + \ 2\Gamma^0_{01} \frac{\mathrm{d}t}{\mathrm{d}\tau} \frac{\mathrm{d}x}{\mathrm{d}\tau} \ + \ \Gamma^0_{11} \left(\frac{\mathrm{d}x}{\mathrm{d}\tau} \right)^2 \\
& + \ \Gamma^0_{22} \left(\frac{\mathrm{d}y}{\mathrm{d}\tau} \right)^2 \ + \ \Gamma^0_{33} \left(\frac{\mathrm{d}z}{\mathrm{d}\tau} \right)^2
\end{aligned}
\tag{57}
$$

Again, Christoffel symbols are taken from section 6. Using constraints (43) and (42) we get:

$$
\begin{aligned}
\Gamma^0_{00} & \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 \ + \ 2\Gamma^0_{01} \frac{\mathrm{d}t}{\mathrm{d}\tau} \frac{\mathrm{d}x}{\mathrm{d}\tau} \ + \ \Gamma^0_{11} \left(\frac{\mathrm{d}x}{\mathrm{d}\tau} \right)^2 \\
& = \ \frac{1}{2} t \left(W_t \frac{\mathrm{d}t}{\mathrm{d}\tau} + W_x \frac{\mathrm{d}x}{\mathrm{d}\tau} \right)^2 - \frac{1}{4t} \left[\left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}\tau} \right)^2 \right] \\
& \quad - \frac{t}{4} (W_t{}^2 - W_x{}^2) \left[\left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 - \left(\frac{\mathrm{d}x}{\mathrm{d}\tau} \right)^2 \right]
\end{aligned}
$$

The first term on the right hand side is nonnegative and will be neglected. Using condition (49) in the form

$$
\left(\frac{\mathrm{d}x}{\mathrm{d}\tau} \right)^2 \ = \ \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 \ + \ g^{00} \left[\varepsilon + g_{22} \left(\frac{\mathrm{d}y}{\mathrm{d}\tau} \right)^2 + g_{33} \left(\frac{\mathrm{d}z}{\mathrm{d}\tau} \right)^2 \right]
$$

21

where as before $\varepsilon \in \{0, 1\}$ depending on causal character of geodesic, we get

$$\Gamma^0_{00} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 + 2\Gamma^0_{01} \frac{\mathrm{d}t}{\mathrm{d}\tau} \frac{\mathrm{d}x}{\mathrm{d}\tau} + \Gamma^0_{11} \left(\frac{\mathrm{d}x}{\mathrm{d}\tau} \right)^2 \geq -\frac{1}{2t} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2$$

$$+ \left[-\frac{1}{4t} + \frac{t}{4}(W_t{}^2 - W_x{}^2) \right] g^{00} \left[\varepsilon + g_{22} \left(\frac{\mathrm{d}y}{\mathrm{d}\tau} \right)^2 + g_{33} \left(\frac{\mathrm{d}z}{\mathrm{d}\tau} \right)^2 \right].$$

This estimate yield for the factor $(2t^{-1})$ of geodesic equation (57)

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[t^{-1} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 \right] \leq$$

$$-g^{00} \left[-\frac{1}{2t^2} + \frac{1}{2}(W_t{}^2 - W_x{}^2) \right] \left[\varepsilon + g_{22} \left(\frac{\mathrm{d}y}{\mathrm{d}\tau} \right)^2 + g_{33} \left(\frac{\mathrm{d}z}{\mathrm{d}\tau} \right)^2 \right]$$

$$+ g^{00} \left[\partial_t g_{22} \cdot t^{-1} \left(\frac{\mathrm{d}y}{\mathrm{d}\tau} \right)^2 + \partial_t g_{33} \cdot t^{-1} \left(\frac{\mathrm{d}z}{\mathrm{d}\tau} \right)^2 \right], \tag{58}$$

where we have used (51).

Due to (50) the scalar products $g(\dot{\mathfrak{r}}, \cdot)$ in

$$\left(\frac{\mathrm{d}y}{\mathrm{d}\tau} \right)^2 = \left[g^{22} \cdot g \left(\dot{\mathfrak{r}}, \frac{\partial}{\partial y} \right) \right]^2$$

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\tau} \right)^2 = \left[g^{33} \cdot g \left(\dot{\mathfrak{r}}, \frac{\partial}{\partial z} \right) \right]^2$$

are constant along the geodesic. Thanks to lemmas 18 and 19 we can estimate all brackets on the right hand side of the inequality (58) by an $\mathcal{O}(t^C)$-term, where the constant C only depends on the model parameter β under consideration. Furthermore, due to (47) it holds $g^{00} \sim \mathcal{O}(\mathrm{e}^{-2\bar{\nu}t})$, $\bar{\nu} > 0$, g^{00} will dominate the development on the right hand side. Consequently there are positive constants C_1, C_2 with

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[t^{-1} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 \right] \leq C_1 \mathrm{e}^{-C_2 t}$$

and furthermore some positive constant $C_3{}^{-2}$, such that for sufficiently large t it holds

$$t^{-1} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 \leq C_3{}^{-2}$$

and a final integration gives

$$\tau(t) \geq \tau(t_1) + 2C_3 \left(\sqrt{t} - \sqrt{t_1} \right),$$

22

which proves $\tau \to \infty$ for $t \to \infty$. $\blacksquare$

We summarize as follows:

Corollary 21 *Every inextendable, future directed, nonspacelike geodesic in an arbitrary polarized Gowdy manifold is in its future direction complete.*

6 Nonvanishing Christoffel symbols in polarized Gowdy spacetimes

In polarized Gowdy models the nonvanishing components of the metric tensor are

$$
\begin{aligned}
g_{00} &= -\mathrm{e}^{2a} & \qquad g_{22} &= t\,\mathrm{e}^{W} \\
g_{11} &= \mathrm{e}^{2a} & \qquad g_{33} &= t\,\mathrm{e}^{-W}
\end{aligned}
$$

where $a = a(t,x)$, $W = W(t,x)$, so up to symmetry the only nonvanishing symbols are

$$
\begin{aligned}
\Gamma^0_{00} &= \tfrac{1}{2}g^{00}\partial_t g_{00} = a_t & \qquad \Gamma^0_{11} &= -\tfrac{1}{2}g^{00}\partial_t g_{11} = a_t \\
\Gamma^0_{01} &= \tfrac{1}{2}g^{00}\partial_x g_{00} = a_x & \qquad \Gamma^1_{01} &= \tfrac{1}{2}g^{11}\partial_t g_{11} = a_t \\
\Gamma^1_{00} &= -\tfrac{1}{2}g^{11}\partial_x g_{00} = a_x & \qquad \Gamma^1_{11} &= \tfrac{1}{2}g^{11}\partial_x g_{11} = a_x
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^0_{22} &= -\tfrac{1}{2}g^{00}\partial_t g_{22} = -\tfrac{1}{2}g^{00}(t^{-1}+W_t)g_{22} \\
\Gamma^2_{02} &= \tfrac{1}{2}g^{22}\partial_t g_{22} = \tfrac{1}{2}(t^{-1}+W_t) \\
\Gamma^1_{22} &= -\tfrac{1}{2}g^{11}\partial_x g_{22} = -\tfrac{1}{2}g^{11}W_x g_{22} \\
\Gamma^2_{12} &= \tfrac{1}{2}g^{22}\partial_x g_{22} = \tfrac{1}{2}W_x \\
\Gamma^0_{33} &= -\tfrac{1}{2}g^{00}\partial_t g_{33} = -\tfrac{1}{2}g^{00}(t^{-1}-W_t)g_{33} \\
\Gamma^3_{03} &= \tfrac{1}{2}g^{33}\partial_t g_{33} = \tfrac{1}{2}(t^{-1}-W_t) \\
\Gamma^1_{33} &= -\tfrac{1}{2}g^{11}\partial_x g_{33} = \tfrac{1}{2}g^{11}W_x g_{33} \\
\Gamma^3_{13} &= \tfrac{1}{2}g^{33}\partial_x g_{33} = -\tfrac{1}{2}W_x
\end{aligned}
$$

Acknowledgement

Literature

Chruściel, P.T., J. Isenberg and V. Moncrief (1990) Strong cosmic censorship in polarised Gowdy spacetimes, *Class. Quant. Grav.*, *7*: 1671-1680.

Gowdy R.H. (1974) Vacuum spacetimes with two-parameter spacelike isometry groups and compact invariant hypersurfaces: topology and boundary conditions, *Ann. Phys.*, *83*: 203-241.

Isenberg, J. and V. Moncrief (1990) Asymptotic behavior of the gravitational field and the nature of singularities in Gowdy spacetimes, *Ann. Phys.*, *199*: 84-122.

Moncrief, V. (1981) Global properties of Gowdy spacetimes with $\mathbb{T}^3 \times \mathbb{R}$ topology, *Ann. Phys.*, *132*: 87-107.

Rendall, A.D. (2000) Fuchsian analysis of singularities in Gowdy spacetimes beyond analyticity, *Class. Quant. Grav.*, *17*: 3305-3316.

Ringström, H. (2002) On Gowdy vacuum spacetimes, *preprint*, `gr-qc/0204044`.